Discrete-Time Markov Chains

Topics

- State-transition matrix
- Network diagrams
- Examples: gambler's ruin, brand switching, IRS, craps
- Transient probabilities
- Steady-state probabilities

Discrete – Time Markov Chains

Many real-world systems contain uncertainty and evolve over time.

Stochastic processes (and Markov chains) are probability models for such systems.

A discrete-time stochastic process is a sequence of random variables $X_0, X_1, X_2, \ldots$ typically denoted by $\{ X_n \}$.

Origins: Galton-Watson process → When and with what probability will a family name become extinct?

Components of Stochastic Processes

The state space of a stochastic process is
the set of all values that the X_n's can take.

(we will be concerned with
stochastic processes with a finite # of states)

Time: $n = 0, 1, 2, \ldots$

State: v-dimensional vector, $\mathbf{s} = (s_1, s_2, \ldots, s_v)$

In general, there are m states,

$\mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^m$ or $\mathbf{s}^0, \mathbf{s}^1, \ldots, \mathbf{s}^{m-1}$

Also, X_n takes one of m values, so $X_n \leftrightarrow \mathbf{s}$.

Gambler's Ruin

At time 0 I have X_0 = \$2, and each day I make a \$1 bet. I win with probability p and lose with probability $1-p$. I'll quit if I ever obtain \$4 or if I lose all my money.

State space is $\boldsymbol{S} = \{ 0, 1, 2, 3, 4 \}$

Let X_n = amount of money I have after the bet on day n.

$$\text{So, } X_1 = \begin{cases} 3 \text{ with probabilty } p \\ 1 \text{ with probabilty } 1-p \end{cases}$$

If $X_n = 4$, then $X_{n+1} = X_{n+2} = \cdots = 4$.

If $X_n = 0$, then $X_{n+1} = X_{n+2} = \cdots = 0$.

Markov Chain Definition

A stochastic process $\{X_n\}$ is called a Markov chain if

$$\Pr\{X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i\}$$

$$= \Pr\{X_{n+1} = j \mid X_n = i\}$$ ← transition probabilities

for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

Discrete time means $n \in N = \{0, 1, 2, \ldots\}$.

The future behavior of the system depends only on the current state i and not on any of the previous states.

Stationary Transition Probabilities

$\Pr\{X_{n+1} = j \mid X_n = i\} = \Pr\{X_1 = j \mid X_0 = i\}$ for all n

(They don't change over time)

We will only consider stationary Markov chains.

The one-step transition matrix for a Markov chain with states $\boldsymbol{S} = \{0, 1, 2\}$ is

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$$

where $p_{ij} = \Pr\{X_1 = j \mid X_0 = i\}$

Properties of Transition Matrix

If the state space $\boldsymbol{S} = \{0, 1, \ldots, m-1\}$ then we have

$$\sum_j p_{ij} = 1 \quad \forall\ i \qquad \text{and} \qquad p_{ij} \geq 0 \quad \forall\ i, j$$

(we must go somewhere)

(each transition has probability ≥ 0)

Gambler's Ruin Example

	0	1	2	3	4
0	1	0	0	0	0
1	$1-p$	0	p	0	0
2	0	$1-p$	0	p	0
3	0	0	$1-p$	0	p
4	0	0	0	0	1

Computer Repair Example

- Two aging computers are used for word processing.
- When both are working in morning, there is a 30% chance that one will fail by the evening and a 10% chance that both will fail.
- If only one computer is working at the beginning of the day, there is a 20% chance that it will fail by the close of business.
- If neither is working in the morning, the office sends all work to a typing service.
- Computers that fail during the day are picked up the following morning, repaired, and then returned the next morning.
- The system is observed after the repaired computers have been returned and before any new failures occur.

States for Computer Repair Example

Index	State	State definitions
0	**s** = (0)	No computers have failed. The office starts the day with both computers functioning properly.
1	**s** = (1)	One computer has failed. The office starts the day with one working computer and the other in the shop until the next morning.
2	**s** = (2)	Both computers have failed. All work must be sent out for the day.

Events and Probabilities for Computer Repair Example

Index	Current state	Events	Prob-ability	Next state
0	$\mathbf{s}^0 = (0)$	Neither computer fails.	0.6	$\mathbf{s}' = (0)$
		One computer fails.	0.3	$\mathbf{s}' = (1)$
		Both computers fail.	0.1	$\mathbf{s}' = (2)$
1	$\mathbf{s}^1 = (1)$	Remaining computer does not fail and the other is returned.	0.8	$\mathbf{s}' = (0)$
		Remaining computer fails and the other is returned.	0.2	$\mathbf{s}' = (1)$
2	$\mathbf{s}^2 = (2)$	Both computers are returned.	1.0	$\mathbf{s}' = (0)$

State-Transition Matrix and Network

The events associated with a Markov chain can be described by the $m \times m$ matrix: $\mathbf{P} = (p_{ij})$.

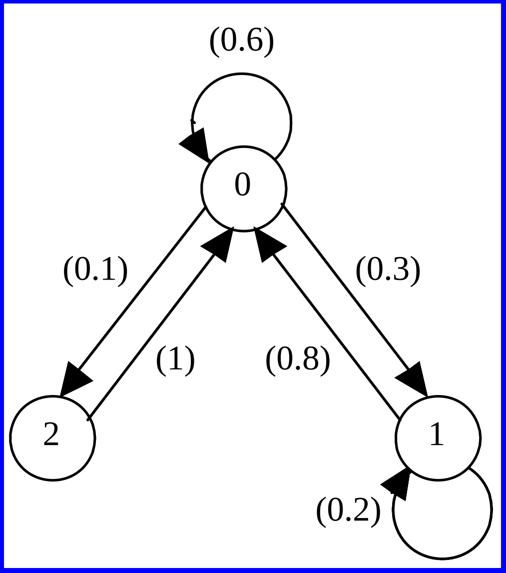
For computer repair example, we have:

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.8 & 0.2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

State-Transition Network

- Node for each state
- Arc from node i to node j if $p_{ij} > 0$.

For computer repair example:



Procedure for Setting Up a DTMC

1. Specify the times when the system is to be observed.
2. Define the state vector **s** = (s_1, s_2, . . . , s_v) and list all the states. Number the states.
3. For each state **s** at time n identify all possible next states **s'** that may occur when the system is observed at time n + 1.
4. Determine the state-transition matrix **P** = (p_{ij}).
5. Draw the state-transition diagram.

Repair Operation Takes Two Days

One repairman, two days to fix computer.

→ new state definition required: $\mathbf{s} = (s_1, s_2)$

s_1 = day of repair of the first machine

s_2 = status of the second machine (working or needing repair)

For s_1, assign
- 0 if 1st machine has not failed
- 1 if today is the first day of repair
- 2 if today is the second day of repair

For s_2, assign
- 0 if 2nd machine has not failed
- 1 if it has failed

State Definitions for 2-Day Repair Times

Index	State	State definitions
0	$s^0 = (0, 0)$	No machines have failed.
1	$s^1 = (1, 0)$	One machine has failed and today is in the first day of repair.
2	$s^2 = (2, 0)$	One machine has failed and today is in the second day of repair.
3	$s^3 = (1, 1)$	Both machines have failed; today one is in the first day of repair and the other is waiting.
4	$s^4 = (2, 1)$	Both machines have failed; today one is in the second day of repair and the other is waiting.

State-Transition Matrix for 2-Day Repair Times

$$\mathbf{P} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \left[\begin{matrix} 0.6 & 0.3 & 0 & 0.1 & 0 \\ 0 & 0 & 0.8 & 0 & 0.2 \\ 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}\right] \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

For example, $p_{14} = 0.2$ is probability of going from state 1 to state 4 in one day, where $\mathbf{s}^1 = (1, 0)$ and $\mathbf{s}^4 = (2, 1)$

Brand Switching Example

Number of consumers switching from brand i in week 26 to brand j in week 27

Brand	(j) 1	2	3	Total
(i)				
1	90	7	3	100
2	5	205	40	250
3	30	18	102	150
Total	125	230	145	500

This is called a contingency table.

→ Used to construct transition probabilities.

Empirical Transition Probabilities for Brand Switching, p_{ij}

Brand	(j) 1	2	3
(i) 1	$\frac{90}{100} = 0.90$	$\frac{7}{100} = 0.07$	$\frac{3}{100} = 0.03$
2	$\frac{5}{250} = 0.02$	$\frac{205}{250} = 0.82$	$\frac{40}{250} = 0.16$
3	$\frac{30}{150} = 0.20$	$\frac{18}{150} = 0.12$	$\frac{102}{150} = 0.68$

Steady state →

Markov Analysis

- State variable, X_n = brand purchased in week n
- $\{X_n\}$ represents a discrete state and discrete time stochastic process, where $\boldsymbol{S} = \{1, 2, 3\}$ and $N = \{0, 1, 2, . . .\}$.
- If $\{X_n\}$ has Markovian property and **P** is stationary, then a Markov chain should be a reasonable representation of aggregate consumer brand switching behavior.

Potential Studies

- Predict market shares at specific future points in time.

- Assess rates of change in market shares over time.

- Predict market share equilibriums (if they exist).

- Evaluate the process for introducing new products.

Transform a Process to a Markov Chain

Sometimes a non-Markovian stochastic process can be transformed into a Markov chain by expanding the state space.

Example: Suppose that the chance of rain tomorrow depends on the weather conditions for the previous two days (yesterday and today).

Specifically,

$\Pr\{\text{rain tomorrow} \mid \text{rain last 2 days (RR)}\} = 0.7$

$\Pr\{\text{rain tomorrow} \mid \text{rain today but not yesterday (NR)}\} = 0.5$

$\Pr\{\text{rain tomorrow} \mid \text{rain yesterday but not today (RN)}\} = 0.4$

$\Pr\{\text{rain tomorrow} \mid \text{no rain in last 2 days (NN)}\} = 0.2$

Does the Markovian Property Hold ?

The Weather Prediction Problem

How to model this problem as a Markov Process ?

The state space: 0 = (RR) 1 = (NR) 2 = (RN) 3 = (NN)

The transition matrix:

		0(RR)	1(NR)	2(RN)	3(NN)
0 (RR)	0.7	0	0.3	0	
P =	1 (NR)	0.5	0	0.5	0
2 (RN)	0	0.4	0	0.6	
3 (NN)	0	0.2	0	0.8	

This is a discrete-time Markov process.

Multi-step (n-step) Transitions

The **P** matrix is for one step: n to $n + 1$.

How do we calculate the probabilities for transitions involving more than one step?

Consider an IRS auditing example:

Two states: $\mathbf{s}^0 = 0$ (no audit), $\mathbf{s}^1 = 1$ (audit)

Transition matrix $\mathbf{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$

Interpretation: $p_{01} = 0.4$, for example, is conditional probability of an audit next year given no audit this year.

Two-step Transition Probabilities

Let $p_{ij}^{(2)}$ be probability of going from *i* to *j* in two transitions.

In matrix form, $\mathbf{P}^{(2)} = \mathbf{P} \times \mathbf{P}$, so for IRS example we have

$$\mathbf{P}^{(2)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix}$$

The resultant matrix indicates, for example, that the probability of no audit 2 years from now given that the current year there was no audit is $p_{00}^{(2)} = 0.56$.

n-Step Transition Probabilities

This idea generalizes to an arbitrary number of steps.

For $n = 3$: $P^{(3)} = P^{(2)}\,P = P^2\,P = P^3$

or more generally, $P^{(n)} = P^{(m)}\,P^{(n-m)}$

The ij th entry of this reduces to

$$p_{ij}^{(n)} = \sum_{k=0}^{m} p_{ik}^{(m)}\,p_{kj}^{(n-m)} \qquad 1 \le m \le n-1$$

Chapman - Kolmogorov Equations

Interpretation:

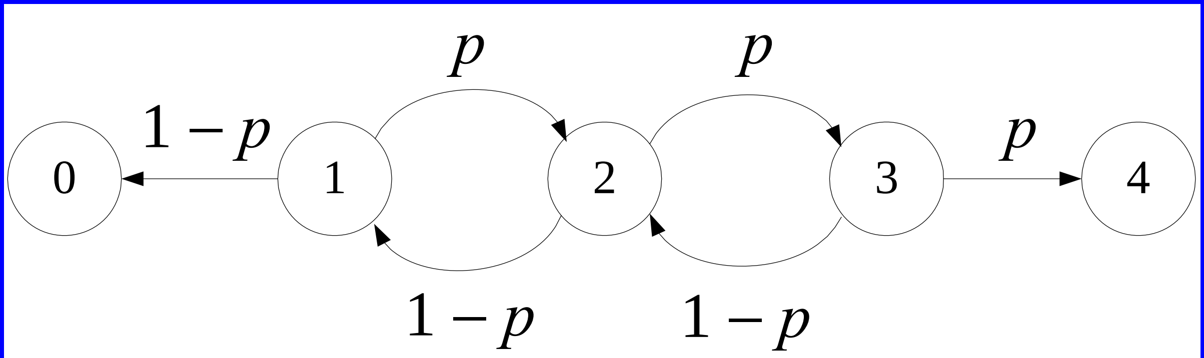
RHS is the probability of going from i to k in m steps & then going from k to j in the remaining $n - m$ steps, summed over all possible intermediate states k.

n-Step Transition Matrix for IRS Example

Time, n	Transition matrix, $\mathbf{P}^{(n)}$
1	$\begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$
2	$\begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix}$
3	$\begin{bmatrix} 0.556 & 0.444 \\ 0.555 & 0.445 \end{bmatrix}$
4	$\begin{bmatrix} 0.5556 & 0.4444 \\ 0.5555 & 0.4445 \end{bmatrix}$
5	$\begin{bmatrix} 0.55556 & 0.44444 \\ 0.55555 & 0.44445 \end{bmatrix}$

Gambler's Ruin Revisited for $p = 0.75$

State-transition network



State-transition matrix

	0	1	2	3	4
0	1	0	0	0	0
1	0.25	0	0.75	0	0
2	0	0.25	0	0.75	0
3	0	0	0.25	0	0.75
4	0	0	0	0	1

Gambler's Ruin with p = 0.75, n = 30

$\mathbf{P}^{(30)} =$

	0	1	2	3	4
0	1	0	0	0	0
1	0.325	ε	0	ε	0.675
2	0.1	0	ε	0	0.9
3	0.025	ε	0	ε	0.975
4	0	0	0	0	1

(ε is very small nonunique number)

What does matrix mean?

A steady state probability does not exist.

DTMC Add-in for Gambler’s Ruin

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**Markov Chain Transition Matrix**													
2	Type:	DTMC			**Step**		Matrix Analyzed.							
3	Title:	ambler_Ruin		**Calculate**	**Measure**		2 Recurrent States							
4			**Change**		Week		2 Recurrent State Classes							
5				**Analyze**			3 Transient States							
6					**State**	5	0	1	2	3	4			
7				**Index**	**Names**		State 0	State 1	State 2	State 3	State 4	**Sum**	**Status**	
8		**Economics**		0	**State 0**	State 0	1	0	0	0	0	1	Class-1	
9				1	**State 1**	State 1	0.25	0	0.75	0	0	1	Transient	
10		**Transient**		2	**State 2**	State 2	0	0.25	0	0.75	0	1	Transient	
11				3	**State 3**	State 3	0	0	0.25	0	0.75	1	Transient	
12		**Steady State**		4	**State 4**	State 4	0	0	0	0	1	1	Class-2	
13						**Sum**	1.25	0.25	1	0.75	1.75			
14		**n-step Probabilities**												
15														
16		**First Pass**												
17														
18		**Simulate**												
19														
20		**Absorbing States**												
21														

30-Step Transition Matrix for Gambler's Ruin

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	**n-Step Transition Matrix**													
2	Type:	DTMC			**30 step Transition Matrix**						**30 step Cost**			
3	Title:	ambler_Ru	Steps/Iter	29	0	1	2	3	4		**Average**		**Present**	
4					State 0	State 1	State 2	State 3	State 4		**per step**		**Worth**	
5		**Start**	0	State 0	1	0	0	0	0		0		0	
6			1	State 1	0.325	2.04E-07	0	6.12E-07	0.674999		0		0	
7		**More**	2	State 2	0.1	0	4.08E-07	0	0.9		0		0	
8			3	State 3	0.025	6.8E-08	0	2.04E-07	0.975		0		0	
9		**Matrix**	4	State 4	0	0	0	0	1		0		0	
10														

Limiting probabilities

	A	B	C	D	E	F	G	H	I	J	K
1	**Absorbing State Ana**			2 absorbing state classes							
2	Type:	DTMC		3 transient states							
3	Title:	ambler_Ruin									
4				Matrix shows long term transition probabilities from transient to absorbi							
5		**Matrix**		Class-1	Class-2						
6				State 0	State 4						
7		Transient State 1		0.325	0.675						
8		Transient State 2		0.1	0.9						
9		Transient State 3		0.025	0.975						
10											

Conditional vs. Unconditional Probabilities

Let state space $\boldsymbol{S} = \{1, 2, \ldots, m\}$.

Let $p_{ij}^{(n)}$ be conditional n-step transition probability → $\mathrm{P}^{(n)}$.

Let $\mathbf{q}(n) = \big(q_1(n), \ldots, q_m(n)\big)$ be vector of all unconditional probabilities for all m states after n transitions.

Perform the following calculations:

$$\mathbf{q}(n) = \mathbf{q}(0)\mathbf{P}^{(n)} \quad \text{or} \quad \mathbf{q}(n) = \mathbf{q}(n-1)\mathbf{P}$$

where $\mathbf{q}(0)$ is initial unconditional probability.

The components of $\mathbf{q}(n)$ are called the transient probabilities.

Brand Switching Example →

We approximate $q_i(0)$ by dividing total customers using brand i in week 27 by total sample size of 500:

$$\mathbf{q}(0) = (125/500, 230/500, 145/500) = (0.25, 0.46, 0.29)$$

To predict market shares for, say, week 29 (that is, 2 weeks into the future), we simply apply equation with $n = 2$:

$$\mathbf{q}(2) = \mathbf{q}(0)\mathbf{P}^{(2)}$$

$$\mathbf{q}(2) = (0.25, 0.46, 0.29)\begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}^2$$

$= (0.327, 0.406, 0.267)$

= expected market share from brands 1, 2, 3

Transition Probabilities for n Steps

Property 1: Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain with state space $\boldsymbol{S}$ and state-transition matrix $\mathbf{P}$. Then for i and $j \in \boldsymbol{S}$, and $n = 1, 2, \ldots$

$$\Pr\{X_n = j \mid X_0 = i\} = p_{ij}^{(n)}$$

where the right-hand side represents the ij^{th} element of the matrix $\mathbf{P}^{(n)}$.

Steady-State Probabilities

Property 2: Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ is the m-dimensional row vector of steady-state (unconditional) probabilities for the state space $\boldsymbol{S} = \{1,\ldots,m\}$. To find steady-state probabilities, solve linear system:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \ \Sigma_{j=1,m} \pi_j = 1, \ \pi_j \geq 0, \ j = 1,\ldots,m$$

Brand switching example:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}$$

$$\pi_1 + \pi_2 + \pi_2 = 1, \ \pi_1 \geq 0, \ \pi_2 \geq 0, \ \pi_3 \geq 0$$

Steady-State Equations for Brand Switching Example

$$\pi_1 = 0.90\pi_1 + 0.02\pi_2 + 0.20\pi_3$$

$$\pi_2 = 0.07\pi_1 + 0.82\pi_2 + 0.12\pi_3$$

$$\pi_3 = 0.03\pi_1 + 0.16\pi_2 + 0.68\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 \geq 0,\ \ \pi_2 \geq 0,\ \ \pi_3 \geq 0$$

Total of 4 equations in 3 unknowns

➔ Discard 3rd equation and solve the remaining system to get :

$$\pi_1 = 0.474,\ \ \pi_2 = 0.321,\ \ \pi_3 = 0.205$$

➔ Recall: $q_1(0) = 0.25,\ \ q_2(0) = 0.46,\ \ q_3(0) = 0.29$

Comments on Steady-State Results

1. Steady-state predictions are never achieved in actuality due to a combination of

   (i) errors in estimating **P**

   (ii) changes in **P** over time

   (iii) changes in the nature of dependence relationships among the states.

2. Nevertheless, the use of steady-state values is an important diagnostic tool for the decision maker.

3. Steady-state probabilities might not exist unless the Markov chain is ergodic.

Existence of Steady-State Probabilities

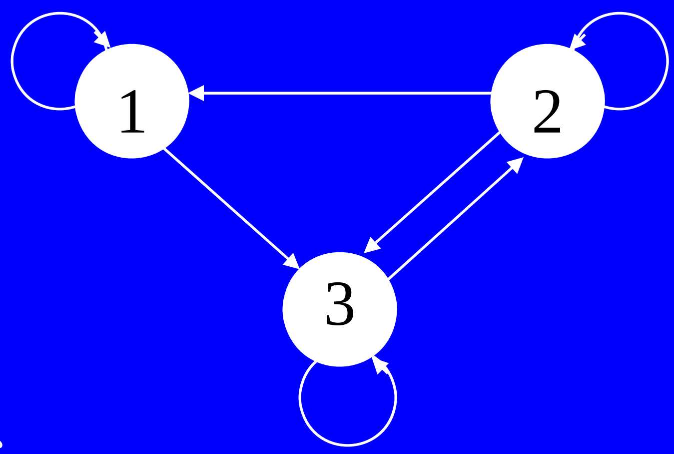
A Markov chain is ergodic if it is aperiodic and allows the attainment of any future state from any initial state after one or more transitions. If these conditions hold, then

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$$

For example,

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0 & 0.9 & 0.1 \end{bmatrix}$$

State-transition network



Conclusion: chain is ergodic.

← Craps

Game of Craps

The game of craps is played as follows. The player rolls a pair of dice and sums the numbers showing.

- Total of 7 or 11 on the first rolls wins for the player
- Total of 2, 3, 12 loses
- Any other number is called the point.

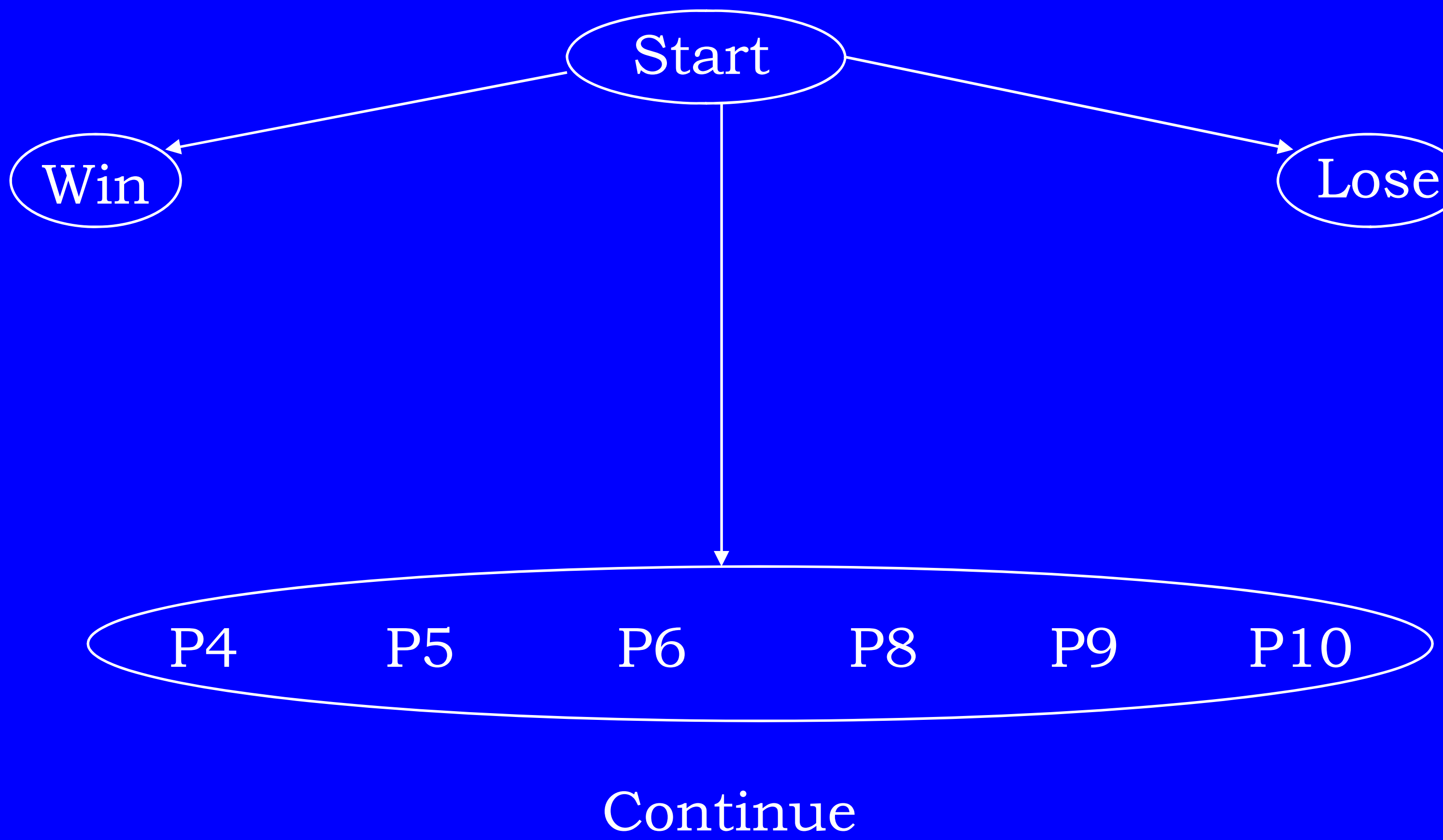
The player rolls the dice again.

- If she rolls the point number, she wins
- If she rolls number 7, she loses
- Any other number requires another roll

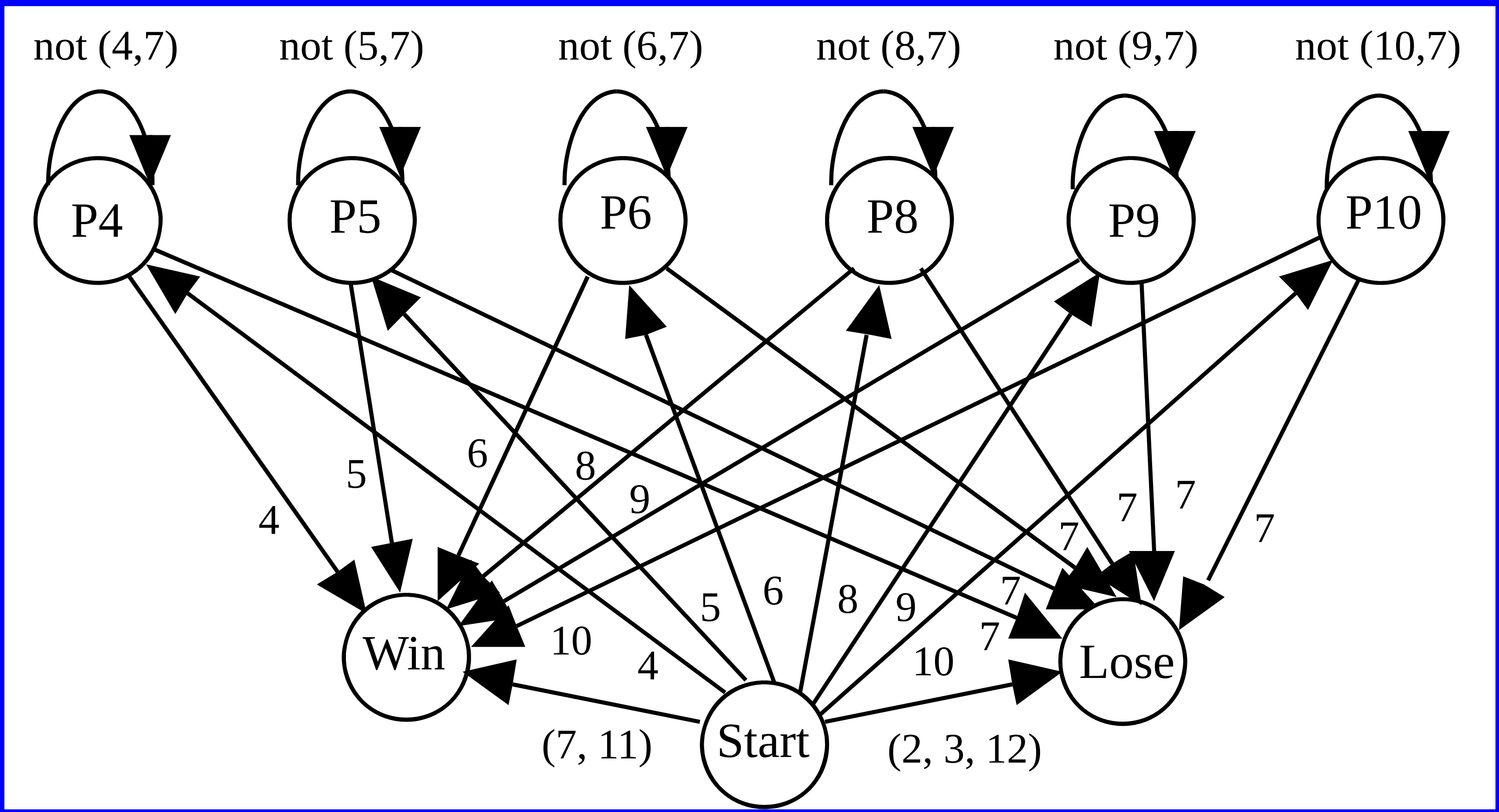
The game continues until he/she wins or loses

Game of Craps as a Markov Chain

All the possible states



Game of Craps Network



Game of Craps

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob.	0.028	0.056	0.083	0.111	0.139	0.167	0.139	0.111	0.083	0.056	0.028

Probability of win = Pr{ 7 or 11 } = 0.167 + 0.056 = 0.223

Probability of loss = Pr{ 2, 3, 12 } = 0.028 + 0.056 + 0.028 = 0.112

P =

	Start	Win	Lose	P4	P5	P6	P8	P9	P10
Start	0	0.222	0.111	0.083	0.111	0.139	0.139	0.111	0.083
Win	0	1	0	0	0	0	0	0	0
Lose	0	0	1	0	0	0	0	0	0
P4	0	0.083	0.167	0.75	0	0	0	0	0
P5	0	0.111	0.167	0	0.722	0	0	0	0
P6	0	0.139	0.167	0	0	0.694	0	0	0
P8	0	0.139	0.167	0	0	0	0.694	0	0
P9	0	0.111	0.167	0	0	0	0	0.722	0
P10	0	0.083	0.167	0	0	0	0	0	0.75

Transient Probabilities for Craps

Roll, *n*	**q**(*n*)	Start	Win	Lose	P4	P5	P6	P8	P9	P10
0	**q**(0)	1	0	0	0	0	0	0	0	0
1	**q**(1)	0	0.222	0.111	0.083	0.111	0.139	0.139	0.111	0.083
2	**q**(2)	0	0.299	0.222	0.063	0.08	0.096	0.096	0.080	0.063
3	**q**(3)	0	0.354	0.302	0.047	0.058	0.067	0.067	0.058	0.047
4	**q**(4)	0	0.394	0.359	0.035	0.042	0.047	0.047	0.042	0.035
5	**q**(5)	0	0.422	0.400	0.026	0.030	0.032	0.032	0.030	0.026

This is not an ergodic Markov chain so where you start is important.

Absorbing State Probabilities for Craps

Initial state	Win	Lose
Start	0.493	0.507
P4	0.333	0.667
P5	0.400	0.600
P6	0.455	0.545
P8	0.455	0.545
P9	0.400	0.600
P10	0.333	0.667

Interpretation of Steady-State Conditions

1. Just because an ergodic system has steady-state probabilities does not mean that the system "settles down" into any one state.

2. The limiting probability π_j is simply the likelihood of finding the system in state *j* after a large number of steps.

3. The probability that the process is in state *j* after a large number of steps is also equals the long-run proportion of time that the process will be in state *j*.

4. When the Markov chain is finite, irreducible and *periodic*, we still have the result that the π_j, $j \in \boldsymbol{S}$, uniquely solve the steady-state equations, but now π_j must be interpreted as the long-run proportion of time that the chain is in state *j*.

What You Should Know About Markov Chains

- How to define states of a discrete time process.
- How to construct a state-transition matrix.
- How to find the n-step state-transition probabilities (using the Excel add-in).
- How to determine the unconditional probabilities after n steps
- How to determine steady-state probabilities (using the Excel add-in).